NORM INEQUALITY FOR GENERALIZED COMMUTATORS OF NORMAL OPERATORS

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Abstract. A commutator of two operators S and T on H is $ST - TS$. We consider A generalized commutator of three operators S, T and R on H

$$STR - RTS$$

Theorem 0.1. let S and R be normal operators with the Cartesian decomposition, $S = A + iC$, $R = B + iD$ such that

$$a_1 \leq A \leq a_2 \text{and} c_1 \leq C \leq c_2$$

$$b_1 \leq B \leq b_2 \text{and} a_1 \leq D \leq d_2,$$

for some real numbers $a_1, a_2, c_1, c_2, b_1, b_2$ and d_1, d_2. Then, for every operator X commuting with S and R,

$$|||SXR - RXS|||\leq \frac{1}{2}\sqrt{(a_1 - a_2)^2 + (c_2 - c_1)^2}\sqrt{(b_1 - b_2)^2 + (d_2 - d_1)^2}\,|||X|||$$

Proof. Let $a = \frac{a_1+a_2}{2}$, $c = \frac{c_1+c_2}{2}$, and $z = a + ib$, also, $b = \frac{b_1+b_2}{2}$, $d = \frac{d_1+d_2}{2}$, and $w = b + id$

$$\begin{aligned} |||SXR - RXS\| &= |||(S - z)X(R - w) - (R - w)X(S - z)\| \\ &\leq \|S - z\|\,|||X(R - w)\| + \|R - w\|\,|||X(S - z)||| \\ &= 2\,\|S - z\|\,\|R - w\|\,|||X||| \end{aligned}$$

But

$$\begin{aligned} \|S - z\|^2 &= \|(A - a) + i(C - c)\|^2 \text{ (by the normality of } S) \\ &\left\|(A - a)^2 + (C - c)^2\right\|^2 \\ &\leq \|(A - a)\|^2 + \|(C - c)\|^2 . \end{aligned}$$

similarly,

$$\|R - w\|^2 \leq \|(B - b)\|^2 + \|(D - d)\|^2$$

Since

$$-\frac{a_2 - a_1}{2} \leq A - a \leq \frac{a_2 - a_1}{2}$$

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and

$$-\frac{c_2 - c_1}{2} \leq C - c \leq \frac{c_2 - c_1}{2}$$

So,

$$\|A - a\| \leq \frac{a_2 - a_1}{2} \text{ and } \|C - c\| \leq \frac{c_2 - c_1}{2}$$

By same way:

$$\|B - b\| \leq \frac{b_2 - b_1}{2} \text{ and } \|D - d\| \leq \frac{d_2 - d_1}{2}$$

$\Rightarrow \|S - z\|^2 \leq (\frac{a_2-a_1}{2})^2 + (\frac{c_2-c_1}{2})^2$ and

$$\|R - w\|^2 \leq (\frac{b_2 - b_1}{2})^2 + (\frac{d_2 - d_1}{2})^2$$

Now, it follows from the inequalities that

$$\| |SXR - RXS\| | \leq \frac{1}{2}\sqrt{(a_1 - a_2)^2 + (c_2 - c_1)^2}\sqrt{(b_1 - b_2)^2 + (d_2 - d_1)^2}\, \| |X\| |$$

□

concerning Theorem . We use the same conditions as in Theorem , so The operator R commuting S and T.

Remark 0.2. R, S and $T \in B(H)$ and $B(H)$ is a Banach algebra, so

$$\|STR - TRS\| = \|STR - TSR\| \leq \|ST - TS\| \, \|R\|$$

Theorem 0.3. By use the above remark and use the same argument in theorem in [kitt], we have

$$\|STR - TRS\| \leq \|ST - TS\| \, \|R\|$$

$$= \|(S - z)(T - w) - (T - w)(S - z)\| \, \|(R - v)\|$$

$$\leq 2 \, \|(S - z)\| \, \|(T - w)\| \, \|(R - v)\|$$

We use v as another complex number with $v = e + if$, and we have by the normality of R

$$\|R - v\|^2 \leq (\frac{e_2 - e_1}{2})^2 + (\frac{f_2 - f_1}{2})^2$$

Thus,

$$\|STR - RTS\| | \leq \frac{1}{4}\sqrt{(a_1 - a_2)^2 + (c_2 - c_1)^2}\sqrt{(b_1 - b_2)^2 + (d_2 - d_1)^2}\sqrt{(e_1 - e_2)^2 + (f_2 - f_1)^2}$$

Acknowledgements

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